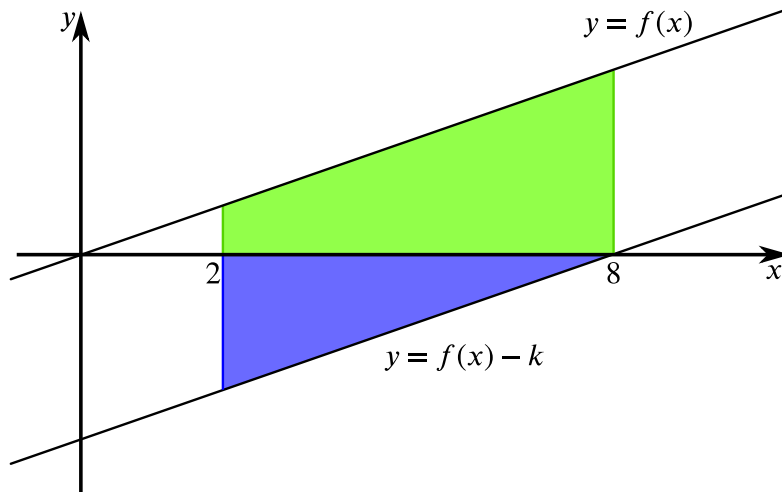


Between the lines

Warm-up

What can you say about this diagram?



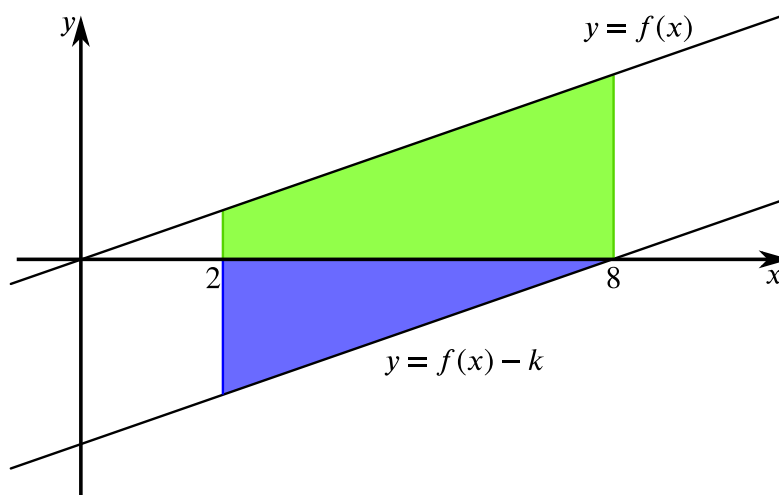
Are there any questions that you would ask about the diagram?

Between the lines

Problem

In the diagram below $f(x)$ is a linear function.

The area of the green shaded region (above the x -axis) is 12 square units.



What is the area of the blue shaded region (below the x -axis)?

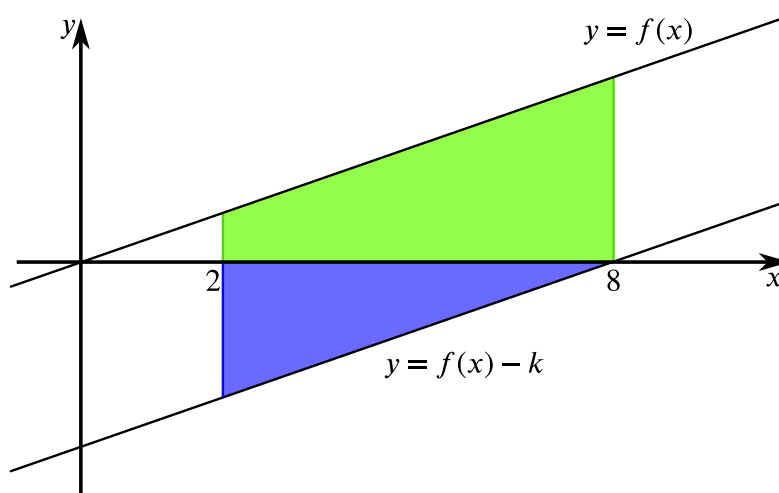
Between the lines

Suggestion



In the diagram below $f(x)$ is a linear function.

The area of the green shaded region (above the x -axis) is 12 square units.



What is the area of the blue shaded region (below the x -axis)?

Take a step back, look at the image and write down anything that you know about it (without doing any calculations).

Some things you might have written

- $f(x)$ passes through the origin.
- $f(x)$ is a straight line.
- $f(x)$ could be written in the form $ax + b$ (where $b = 0$ in this case).
- $y = f(x) - k$ is parallel to $y = f(x)$ (it is a vertical translation by k units downwards).
- The green region is a trapezium of 'height' 6 units.
- The blue region is a triangle of 'base' or 'height' 6 units.
- The combination of the green and blue shaded regions produces a parallelogram.
- The blue triangle could sit 'on top of' the green trapezium to create a rectangle.
- The area of the blue triangle must be less than 12 square units.

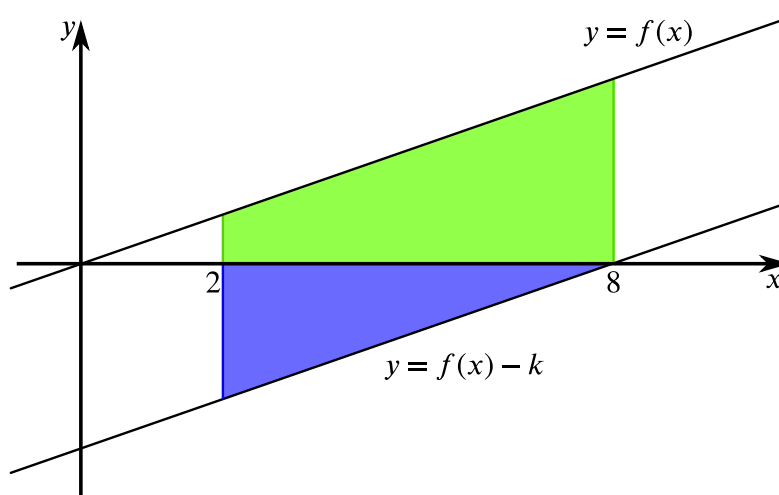
Between the lines

Solution



In the diagram below $f(x)$ is a linear function.

The area of the green shaded region (above the x -axis) is 12 square units.



What is the area of the blue shaded region (below the x -axis)?

There's more than one way in which we could calculate the area of the blue triangle. For most methods that we might consider we will need to identify the value of y when $x = 2$ (let's call this y_1) and when $x = 8$ (let's call this y_2).



How do we know that identifying these two values, y_1 and y_2 , is a good strategy and will provide enough information to solve the problem?

In order to identify these two values we might begin by writing down an equation involving them. This could be the area of the green trapezium which can be written as

$$12 = \frac{1}{2}(y_1 + y_2) \times 6.$$

Rearranging this we find that

$$y_1 + y_2 = 4. \quad (1)$$



In order to identify the unique values of y_1 and y_2 we now need a second equation containing these two unknowns.

We know that $f(x) = ax$ where a is the gradient of the line. Because this is a straight line, the gradient, a , will be the same when $x = 2$ and when $x = 8$ and therefore we can write

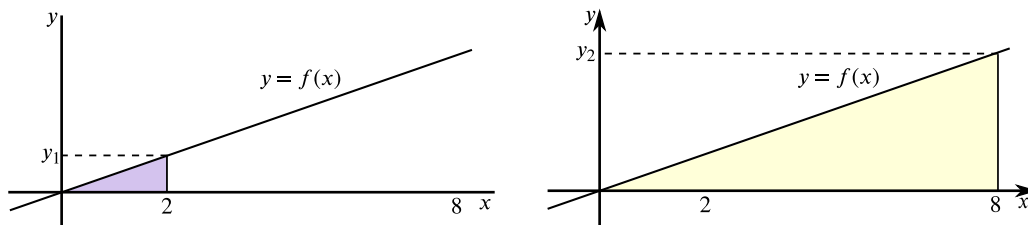
$$a = \frac{y_1}{2} = \frac{y_2}{8},$$

which can be rearranged to give

$$4y_1 = y_2. \quad (2)$$



Notice that we could have obtained this second equation by considering similar triangles instead.



The purple and yellow triangles are similar and related by a linear scale factor of 4. Therefore $y_2 = 4y_1$.

Solving equations (1) and (2) simultaneously we find that $y_1 = \frac{4}{5}$ and $y_2 = \frac{16}{5}$.

Now, the area of the blue triangle can be calculated directly if we first calculate the perpendicular height of the blue triangle as $y_2 - y_1 = \frac{16}{5} - \frac{4}{5} = \frac{12}{5}$. Then we can calculate the area of the blue triangle as

$$\frac{1}{2} \times 6 \times \frac{12}{5} = \frac{36}{5} \text{ square units.}$$



Notice that the area of the blue triangle is less than 12 square units. Is this what you expected?



It is also worth noting that, whilst we didn't need it in order to solve the problem, we now know that

$$y = f(x) = \frac{2}{5}x$$

and that the translated line has equation

$$y = f(x) - k = \frac{2}{5}x - \frac{16}{5},$$

which can also be written as

$$y = \frac{2}{5}(x - 8).$$

An alternative approach

We can also solve the problem without working out the values of y_1 and y_2 explicitly.

Since the lines are a distance of k apart, we have $y_2 = k$. By similar triangles, as above, $y_1 = \frac{1}{4}y_2 = \frac{1}{4}k$.

Therefore the green trapezium, which has 'height' 6 and parallel sides of k and $\frac{1}{4}k$, has area

$$\frac{1}{2} \times 6(k + \frac{1}{4}k) = 3 \times \frac{5}{4}k.$$

The blue triangle also has 'height' 6 but its base is $k - y_1 = \frac{3}{4}k$. Therefore its area is

$$\frac{1}{2} \times 6 \times \frac{3}{4}k = 3 \times \frac{3}{4}k.$$

Therefore the ratio of the green area to the blue area is 5 to 3, and so the blue area is

$$\frac{3}{5} \times 12 = \frac{36}{5}.$$



Reflection

- At first sight, it may not have looked as though there was enough information to solve the problem. Why was it possible?
- How can you generalise the problem?